

# A note on the location of polynomial roots

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## Abstract

We review some known inclusion results for the roots of a polynomial, and adapt them to a conjecture recently presented by S. A. Vavasis. In particular, we provide strict upper and lower bounds to the distance of the closest root of a polynomial  $p(z)$  from a given  $\zeta \in \mathbb{C}$  such that  $p'(\zeta) = 0$ .

## 1 Introduction

Recently S.A. Vavasis [2] has presented the following conjecture.

**Conjecture** *There exist two universal constants  $0 < \iota_1 \leq 1 \leq \iota_2$  with the following property. Let  $\xi_1, \dots, \xi_n$  be the roots of a degree- $n$  univariate polynomial  $p(z)$ . Let  $\zeta_1, \dots, \zeta_{n-1}$  be the roots of its derivative. Define*

$$\rho_j = \min_{k=2, \dots, n} \left| \frac{k!p(\zeta_j)}{p^{(k)}(\zeta_j)} \right|^{1/k}, \quad j = 1, \dots, n-1 \quad (1)$$

and the annuli

$$A_j = \{z : \iota_1 \rho_j \leq |z - \zeta_j| \leq \iota_2 \rho_j\}, \quad j = 1, \dots, n-1.$$

Then for each  $i = 1, \dots, n$

$$\xi_i \in A_1 \cup \dots \cup A_{n-1}.$$

The author also refers to an unpublished communication by Giusti et Al., where it is shown that  $\iota_1$  exists and can be taken  $(\sqrt{5} - 1)/2$  and where a sequence of  $n$ -degree polynomials is given such that  $\lim_n |z - \zeta_j|/\rho_j = +\infty$  so that  $\iota_2$  does not exist.

In this note we revisit some known general bounds to the roots of a polynomial from [1], in particular Theorem 6.4b on pages 451,452, and Theorem 6.4e on page 454, and adapt them to the conditions of the Vavasis conjecture.

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More specifically, we show that for any polynomial  $p(z)$ , and for any  $\zeta$  such that  $p'(\zeta) = 0$ , there exists a root  $\xi$  of  $p(z)$  satisfying

$$|\xi - \zeta| \leq \rho \sqrt{n/2}, \quad \rho = \min_{k=2, \dots, n} \left( \frac{k! p(\zeta)}{p^{(k)}(\zeta)} \right)^{1/k},$$

and that the bound is sharp since it is attained by a suitable polynomial.

We provide also some sharp lower bound to  $|\xi - \zeta|$  under the condition that  $p^{(k)}(\zeta) = 0$  for  $k \in \Omega$ , where  $\Omega$  is a nonempty subset of  $\{1, 2, \dots, n-1\}$ .

Moreover, we also show that  $\iota_2$  does not exist by providing an example of a sequence  $\{p_n(z)\}_n$  of polynomials of degree  $n+1$  having a common root  $\xi$ , where the ratio  $|\xi - \zeta_j^{(n)}|/\rho_j^{(n)}$  is independent of  $j$  and tends to infinity as  $n^{1-\epsilon}$  for any  $i = 1, \dots, n$  and for any  $0 < \epsilon < 1$ , where  $\zeta_j^{(n)}$  are the roots of  $p'_n(z)$ .

## 2 Main results

In this section, after providing a counterexample of the Vavasis conjecture, we review some inclusion theorems of [1], which give lower bounds and upper bounds to the distance of the roots of a polynomial from a given complex number  $\zeta$ .

### 2.1 Counterexample

Consider the monic polynomial of degree  $n+1$

$$p_n(z) = z^{n+1} - (n+1)z.$$

Clearly  $z = 0$  is one of its roots, and we have  $p'_n(z) = (n+1)(z^n - 1)$ , so that the roots  $\zeta_i$  of  $p'_n$  are the complex  $n$ -th roots of the unity. Define

$$\rho^{(n,k)} = \left| \frac{k! p_n(\zeta)}{p_n^{(k)}(\zeta)} \right|^{1/k}, \quad \rho^{(n)} = \min_{k=2, \dots, n+1} \rho^{(n,k)},$$

where  $\zeta$  stands for any  $n$ -th root  $\zeta_i$  of 1, and observe that  $p_n(\zeta) = -n\zeta$ ,  $p_n^{(2)}(\zeta) = n(n+1)\zeta^{-1}$ . Therefore, for  $k = 2$  one has

$$\rho^{(n)} \leq \rho^{(n,2)} = \left| \frac{2! p_n(\zeta)}{p_n^{(2)}(\zeta)} \right|^{1/2} = \left| \frac{2n}{n(n+1)} \right|^{1/2} = \sqrt{\frac{2}{n+1}}$$

hence  $\rho_n \rightarrow 0$  as  $n \rightarrow \infty$ . Observe that this bound is independent of the root  $\zeta_i$ . The annuli  $A_i$  have their centers on the unit circle and for  $\iota_2$  constant, their external radii tend to 0 as  $n \rightarrow \infty$ . Thus, for sufficiently large values of  $n$  they cannot contain the origin, and this contradicts the conjecture as  $z = 0$  is a common root to all the polynomials  $p_n(z)$ .

Moreover, for  $z = 0$  one has

$$\frac{|z - \zeta_i|}{\rho^{(n)}} \geq \frac{|z - \zeta_i|}{\rho^{(n,2)}} = (n+1)^{1/(n+1)} \sqrt{\frac{n+1}{2}} \geq \sqrt{\frac{n+1}{2}}.$$

That is, the ratio  $\frac{|z-\zeta_i|}{\rho^{(n,k)}}$  can grow as much as  $\sqrt{n/2}$ . For general  $k$  one can easily get

$$\frac{|z-\zeta|}{\rho^{(n,k)}} = \left[ \frac{1}{n} \binom{n+1}{k} \right]^{1/k} (n+1)^{\frac{1}{n+1}} \geq \left[ \frac{1}{n} \binom{n+1}{k} \right]^{1/k}. \quad (2)$$

Thus, for a fixed  $k$  the ratio  $|z-\zeta|/\rho^{(n,k)}$  can grow as much as  $n^{1-\frac{1}{k}}$ .

## 2.2 Lower bounds

Let us recall the following result (see [1], Theorem 6.4b).

**Theorem 1** *Let  $p(z) = \sum_{i=0}^n a_i z^i$  be a monic polynomial of degree  $n$  and  $\zeta$  any complex number. Assume  $a_0 \neq 0$ . Then any root  $\xi$  of  $p(x)$  is such that*

$$\gamma\rho < |\xi - \zeta|, \quad \rho = \rho(\zeta) = \min_{k=2,\dots,n} \left| k! \frac{p(\zeta)}{p^{(k)}(\zeta)} \right|^{1/k} \quad (3)$$

where  $\gamma = 1/2$ .

The following proof of the above theorem can be easily adjusted to the case where  $\zeta$  is a (numerical) root of some derivative of  $p(z)$ .

Without loss of generality we may assume  $\zeta = 0$ . In fact, if  $\zeta \neq 0$  consider  $\hat{p}(z) = p(z - \zeta)$  so that  $\hat{p}'(z) = p'(z - \zeta)$  and  $\hat{\rho}(0) = \rho(\zeta)$ , and reduce the case to  $\zeta = 0$ .

From the definition of  $\rho$  one has

$$\rho^k \leq k! \left| \frac{p(0)}{p^{(k)}(0)} \right| = \left| \frac{a_0}{a_k} \right|. \quad (4)$$

Then taking the moduli in both sides of the equation  $-a_0 = a_1\xi + a_2\xi^2 + \dots + a_n\xi^n$  yields

$$1 \leq \sum_{i=1}^n \left| \frac{a_i}{a_0} \xi^i \right|$$

which, in view of (4) provides the bound

$$1 \leq \sum_{i=1}^n t^i, \quad t = \frac{|\xi|}{\rho},$$

whence

$$1 \leq \frac{t - t^{n+1}}{1 - t}.$$

If  $t < 1$  then we have  $1 - t \leq t - t^{n+1} < t$  which implies  $t > 1/2$ . This proves the bound  $|\xi| > \frac{1}{2}\rho$  for any root  $\xi$  of  $p(z)$ .

Observe that the bound is strict since the polynomial  $p_n(z) = \sum_{i=1}^n z^i - 1$  has a root in the interval  $(1/2, 1/2(1 + 1/n))$  for  $n \geq 2$ .

The proof of Theorem 1 can be adjusted to the case where  $\zeta$  satisfies some additional condition. We have the following result:

**Proposition 1** Assume that  $\zeta$  satisfies the following condition

$$\theta^i \left| \frac{p^{(i)}(\zeta)}{i!p(\zeta)} \right| \leq \epsilon, \quad i \in \Omega = \{i_1, \dots, i_h\} \subset \{1, 2, \dots, n-1\}$$

where  $0 \leq \epsilon < 1/h$ ,  $1 \leq h < n$  and  $\theta$  is an upper bound to  $|\zeta - \xi_i|$  for  $i = 1, \dots, n$ . Then (3) holds where  $\gamma$  is the only solution in  $(1/2, 1)$  of the equation

$$(t-1) \sum_{i \in \Omega} t^i + 2t - 1 + (1-t)h\epsilon = 0. \quad (5)$$

**Proof.** By following the same arguments of the proof of Theorem 1 with  $\zeta = 0$  one obtains

$$1 \leq \sum_{i=1}^n \left| \frac{a_i}{a_0} \xi^i \right| \leq \sum_{i=1, n; i \notin \Omega} \left| \frac{a_i}{a_0} \xi^i \right| + h\epsilon \leq \sum_{i=1, n; i \notin \Omega} t^i + h\epsilon.$$

If  $t < 1$ , replacing  $\sum_{i=1, n; i \notin \Omega} t^i = (t - t^{n+1})/(1-t) - \sum_{i \in \Omega} t^i$  in the latter inequality yields  $1-t \leq t - t^{n+1} - (1-t) \sum_{i \in \Omega} t^i + (1-t)h\epsilon \leq t + (t-1) \sum_{i \in \Omega} t^i + (1-t)h\epsilon$ . Whence,  $t > \gamma$  where  $\gamma$  is the only solution of (5) in  $(1/2, 1)$ .  $\square$

Let us look at some specific instances of the above result. For  $\epsilon = 0$  the condition of the proposition turns into  $p^{(i)}(\zeta) = 0$  for  $i \in \Omega$ . If in addition  $\Omega = \{1\}$  one finds the condition  $p'(\zeta) = 0$  of the Vavasis conjecture and (5) turns into  $t^2 + t - 1 = 0$  that implies  $\gamma = (\sqrt{5} - 1)/2 = 0.618\dots$ . Weaker bounds are obtained assuming  $\epsilon = 0$  and  $\Omega = \{k\}$  for some  $k > 1$  since the only root of the polynomial  $t^{k+1} - t^k + 2t - 1$  in  $(1/2, 1)$  is lower than  $(\sqrt{5} - 1)/2$ .

Better bounds are obtained if  $\zeta$  is a root of multiplicity  $h$  of  $p'(z)$ ; in fact,  $\gamma$  is the only positive root of the polynomial  $t^{h+1} + t - 1$ . In particular, if  $h = 2$  then  $\gamma = 0.682\dots$ , if  $h = 3$ ,  $\gamma = 0.724\dots$ .

If  $\zeta$  is close to a root of  $p'(z)$ , so that the condition  $\theta|p'(\zeta)/p(\zeta)| < \epsilon$  for some “small”  $\epsilon$  is satisfied, then  $\gamma = (\sqrt{5} - 1)/2 - \epsilon(1 + 3/\sqrt{5}) + O(\epsilon^2)$ .

For  $\epsilon = 0$  the bound in the above proposition is strict since it is asymptotically attained by the polynomial  $t^n - (t-1) \sum_{i \in \Omega} t^i - 2t + 1$ . The advantage of this bound is that it allows to compute sharper values for  $\gamma$  just by solving a low degree equation if  $\Omega$  is made up by small integers.

Slightly better lower bounds can be obtained from the following known result of [1] which requires to compute a positive root of a polynomial of degree  $n$ .

**Theorem 2** Any root  $\xi$  of  $p(z)$  is such that  $|\xi| \geq \sigma$ , where  $\sigma$  is the only positive solution to the equation  $|a_0| = \sum_{i=1}^n t^i |a_i|$ .

## 2.3 Upper bounds

Throughout this section we denote

$$\rho^{(k)} = \left( k!p(\zeta)/p^{(k)}(\zeta) \right)^{1/k}, \quad \rho = \min_k \rho^{(k)}$$

for a given  $\zeta \in \mathbb{C}$ . Concerning upper bounds to the distance of a root from  $\zeta$  we recall the following result of [1] (Theorem 6.4e, page 454).

**Theorem 3** *For any  $\zeta \in \mathbb{C}$  there exists a root  $\xi$  of  $p(z)$  such that*

$$|\xi - \zeta| \leq \rho^{(k)} \binom{n}{k}^{1/k}, \quad k = 1, \dots, n. \quad (6)$$

Observe that, for  $k = 2$  one has

$$|\xi - \zeta| \leq \rho^{(2)} \sqrt{n(n-1)/2}, \quad (7)$$

while

$$|\xi - \zeta| \leq \min_k \binom{n}{k}^{1/k} \rho^{(k)} \leq \max_k \binom{n}{k}^{1/k} \rho \leq n\rho. \quad (8)$$

The bound (8) is sharp since it is attained by the polynomial  $p(z) = (z - n)^n$  with  $\zeta = 0$ . In fact, it holds  $\rho = \rho^{(1)} = 1$  and  $p(z)$  has roots of modulus  $n$ .

Under the condition  $p'(\zeta) = 0$  the bounds (6), (7) and (8) can be substantially improved. In fact we may prove the following result

**Proposition 2** *For any  $\zeta \in \mathbb{C}$  such that  $p'(\zeta) = 0$  there exists a root  $\xi$  of  $p(z)$  such that*

$$|\xi - \zeta| \leq \begin{cases} \rho^{(2)} \sqrt{n/2} \\ \rho^{(3)} \sqrt[3]{n/3} \\ \rho^{(k)} \sqrt{n} \left( \frac{1}{k} \prod_{i=2}^{\lfloor k/2 \rfloor} \left( \frac{1}{n} + \frac{1}{2i-1} + \frac{1}{2i-2} \right) \right)^{1/k} \end{cases} \quad \text{for } 4 \leq k \leq n \quad (9)$$

Moreover,

$$|\xi - \zeta| \leq \rho \sqrt{\frac{n}{2}} \quad (10)$$

**Proof.** Without loss of generality we may assume  $\zeta = 0$  and  $a_0 = 1$  so that the polynomial can be written as  $p(z) = 1 + a_2 z^2 + \dots + a_n z^n$ . Recall the Newton identities [1], page 455:

$$ka_k = -s_k - \sum_{i=1}^{k-1} a_i s_{k-i}, \quad k = 1, 2, \dots,$$

where  $s_k = \sum_{i=1}^n \xi_i^{-k}$  are the power sums of the reciprocal of the roots  $\xi$  of  $p(z)$ . Clearly,  $a_1 = s_1 = 0$  so that for  $k \geq 4$  the Newton identities turn into

$$ka_k = -s_k - \sum_{i=2}^{k-2} a_i s_{k-i}, \quad k = 4, 5, \dots \quad (11)$$

Let  $\Delta = \min_i |\xi_i|$  so that  $|s_k| \leq n\Delta^{-k}$ . It holds  $|2a_2| = |s_2| \leq n\Delta^{-2}$ ,  $|3a_3| = |s_3| \leq n\Delta^{-3}$  and

$$k|a_k| \leq \Delta^{-k} n \left(1 + \sum_{i=2}^{k-2} |a_i| \Delta^i\right), \quad k \geq 4.$$

Denoting  $\gamma_k = n(1 + \sum_{i=2}^{k-2} |a_i| \Delta^i)$ , for  $k \geq 4$  and  $\gamma_2 = \gamma_3 = n$ , by using the induction argument one easily finds that

$$\begin{aligned} k|a_k| &\leq \Delta^{-k} \gamma_k \\ \gamma_k &\leq \gamma_{k-1} + \frac{n}{k-2} \gamma_{k-2}, \quad k \geq 4 \\ \gamma_2 &= \gamma_3 = n. \end{aligned} \tag{12}$$

The above expression provides the bound

$$\Delta \leq \rho^{(k)} \left( \frac{\gamma_k}{k} \right)^{1/k} \tag{13}$$

so that it remains to give upper bounds to  $\gamma_k$ . Since  $\gamma_2 = \gamma_3 = n$ , from (13) we deduce (9) for  $k = 2, 3$ . For the general case  $k \geq 4$ , we express the recurrence (12) in matrix form as

$$\begin{bmatrix} \gamma_{k+1} \\ \gamma_k \end{bmatrix} \leq \begin{bmatrix} 1 & \frac{n}{k-1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \gamma_k \\ \gamma_{k-1} \end{bmatrix},$$

where the inequality holds component-wise. Applying twice the above bound yields

$$\begin{bmatrix} \gamma_{k+1} \\ \gamma_k \end{bmatrix} \leq \begin{bmatrix} 1 + \frac{n}{k-1} & \frac{n}{k-2} \\ 1 & \frac{n}{k-2} \end{bmatrix} \begin{bmatrix} \gamma_{k-1} \\ \gamma_{k-2} \end{bmatrix}. \tag{14}$$

Whence, since  $\gamma_2 = \gamma_3 = n$ , one finds that  $\gamma_{2i}$  and  $\gamma_{2i+1}$  are polynomials in  $n$  of degree  $i$ . Denoting

$$\gamma_{2i} = n^i \delta_{2i}, \quad \gamma_{2i+1} = n^i \delta_{2i+1}, \tag{15}$$

we may give upper bounds to  $\delta_k$ . In fact, from (14) with  $k = 2i$  it holds

$$\begin{bmatrix} \delta_{k+1} \\ \delta_k \end{bmatrix} \leq \begin{bmatrix} \frac{1}{n} + \frac{1}{k-1} & \frac{1}{k-2} \\ \frac{1}{n} & \frac{1}{k-2} \end{bmatrix} \begin{bmatrix} \delta_{k-1} \\ \delta_{k-2} \end{bmatrix}. \tag{16}$$

Let us denote  $W_k$  the matrix in the right-hand side of (16), so that for  $n \geq 4$  we have

$$\begin{bmatrix} \delta_{2i+1} \\ \delta_{2i} \end{bmatrix} = W_{2i} W_{2(i-1)} \cdots W_4 \begin{bmatrix} \delta_3 \\ \delta_2 \end{bmatrix}. \tag{17}$$

Since for  $n \geq 4$  we have  $\|W_k\|_\infty = \frac{1}{n} + \frac{1}{k-1} + \frac{1}{k-2}$ , taking norms in (17) yields

$$\|(\delta_{2i+1}, \delta_{2i})\|_\infty \leq \prod_{j=2}^i \|W_{2j}\|_\infty \|(\delta_3, \delta_2)\|_\infty \leq \prod_{j=2}^i \left( \frac{1}{n} + \frac{1}{2j-1} + \frac{1}{2j-2} \right),$$

since  $\|(\delta_3, \delta_2)\|_\infty = \|(1, 1)\|_\infty = 1$ . In view of (13) and (15) this proves (9).

In order to prove the bound (10), from (13) it is sufficient to prove that

$$\gamma_k \leq k \left( \sqrt{\frac{n}{2}} \right)^k. \quad (18)$$

We prove the latter bound by induction on  $k$  for  $2 \leq k \leq n$ . For  $k = 2, 3$ , the inequality (18) is true since  $\gamma_2 = \gamma_3 = n$ . Moreover, from (12) one has  $\gamma_4 \leq \gamma_3 + \frac{n}{2}\gamma_2 = n(n+2)/2$  so that (18) is satisfied also for  $k = 4$ . Now we assume that the bound (18) is true for  $k$  and  $k-1$ , where  $k \geq 4$  and we prove it for  $k+1 \leq n$ , i.e.,  $\gamma_{k+1} \leq (k+1)(\sqrt{n/2})^{k+1}$ . From (12) and from the inductive assumption one has  $\gamma_{k+1} = (\sqrt{\frac{n}{2}})^{k+1} \left( k\sqrt{\frac{2}{n}} + 2 \right)$ . Therefore it is sufficient to prove that  $k\sqrt{\frac{2}{n}} + 2 \leq k+1$ , that is,  $\sqrt{\frac{n}{2}} \geq \frac{k}{k-1}$  which is satisfied for  $n \geq k \geq 4$ . This completes the proof.  $\square$

Observe that the bound of Theorem 2 is sharp since it is attained by the polynomial  $p(z) = (z^2 - m)^m$  with  $\zeta = 0$ , where  $n = 2m$ . In fact,  $p'(0) = 0$ ,  $\rho = \rho^{(2)} = 1$  and the roots of  $p(z)$  have moduli  $\sqrt{n/2}$ .

If  $\zeta$  is such that  $p^{(j)}(\zeta) = 0$ ,  $j = 1, \dots, h$ , then from the Newton identities one finds that  $s_i = a_i = 0$ ,  $i = 1, \dots, h$  so that equation (11) turns into

$$ka_k = -s_k - \sum_{i=h+1}^{k-h-1} a_i s_{k-i}, \quad k \geq 2(h+1).$$

By following the same argument used in the proof of Proposition 2 we can prove that there exists a root  $\xi$  of  $p(z)$  such that

$$|\xi - \zeta| \leq \rho^{(h+i)} \sqrt[h+i]{\frac{n}{h+1}}, \quad i = 1, \dots, h+1.$$

## References

- [1] P. Henrici, *Applied and Computational Complex Analysis*, Vol. 1, Wiley, 1974.
- [2] S. A. Vavasis, A conjecture that the roots of a univariate polynomial lie in a union of annuli (Interim Revised Version), arXiv:math.CV/0606194 v3, 28 Jul 2006.